The Cryosphere Discussions

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Reply on RC1

Noriaki Ohara et al.

Author comment on "A new Stefan equation to characterize the evolution of thermokarst lake and talik geometry" by Noriaki Ohara et al., The Cryosphere Discuss., https://doi.org/10.5194/tc-2021-192-AC1, 2021

Thank you very much for reviewing our manuscript. We are happy to learn your positive evaluation highlighted by the first sentence, "approach is relatively novel, a mathematical model used to represent this process is a good idea". Here, we provide responses to a few key points to enhance the information available to reviewers during the discussion round. The first point is on the validation:

- Are the results sufficient to support the interpretations and conclusions?
  Not really… as noted above, the model derivation is nice, but comparing the model results to one measured lake is a little worrying.

There are only a very limited number of talik depth measurements under an isolated lake in a continuous permafrost – many of these examples are single drill points. The example at Peatball lake is, to our knowledge, the only quasi-3D dataset available in the Arctic. We strongly believe that the TEM sounding survey of the Peatball Lake is the most comprehensive dataset, and therefore most appropriate for this comparison.

- Some minor issues noted below, and some jargon and unnecessarily complex language used to describe especially mathematical derivations.
- Eq 13 I think it may make more logical sense to present this in the opposite direction - the integral along the phase boundary (line) is not something that I can interpret easily or can be visualized, whereas something more like the flux across the phase change surface, or the volume integral of the total energy in the lake is more easily interpreted. I would start with the resulting equation and state which theorem (Stoke's?) is used to get the first equation. Importantly providing a physical interpretation (in more simple terms) of what each expression (start and derived result) means and how it is useful and what it tells us about the system. This would greatly increase the utility of the work for those who are less interested in the mathematics and more interested in their application.

Unfortunately, as this study covers the different fields of studies, some jargon is unavoidable – indeed, we will work to improve this if given the opportunity to submit a revision. We would like to highlight that the methodology itself is novel in many aspects. We propose more explanation in the upcoming revision stage so that physicists, earth scientists as well as mathematicians can understand the content better. One possible reason for confusion may be too many appearances of "Euler" and "Lagrange" in terminology across the related fields with slight variations.

For the sake of simplicity, the original manuscript focused on the mathematical technique, which appears as “Euler equation in the calculus of variation”. However, we propose to enhance explanation with the physical context beyond the Newtonian mechanics, which hopefully helps readers can understand the methodological background of this study. Additionally, the keywords are available in the Wikipedia, which provides fairly accurate explanations usually in plain language for readers who do not have a proper background.

https://en.wikipedia.org/wiki/Lagrangian_mechanics

“Euler–Lagrange equation” may replace “Euler equation in the calculus of variation”.

https://en.wikipedia.org/wiki/Euler%E2%80%93Lagrange_equation

We propose the following additional paragraph placed at the beginning of Chapter 2 Theory.

“This study uses the stationary action principle (the principle of least action) based on the Lagrangian mechanics, which generalizes the classical Newtonian mechanics. The action is defined as the integral of the Lagrangian, which consists of kinetic and potential energies of the system. In this permafrost application, the Lagrangian simply becomes the potential energy due to absence of kinetic energy. The variational principle, the main tool in Lagrangian mechanics, can indeed derive the equations in the Newtonian mechanics. One of the related research topics using such a variational principle is a phase boundary propagation that can be analyzed by the phase field model or diffusion-interface model (Cahn and Hilliard, 1958; Cassel, 2013). This model explains the diffuse phase boundary without surface tension, which appears in Newtonian interfacial physics between liquid and gas but irrelevant for liquid-solid interface. According to the second law of thermodynamics, monotonical decrease of the free energy is required for a non-negative entropy production (Singer-Loginova and Singer, 2008). This requires the time rate of change of the phase boundary to be expressed by the functional derivative of the free energy functional, which corresponds to the basin integrated energy flux in the permafrost thaw problem. This study directly and analytically solves the Euler-Lagrange equation based on the stationary action principle rather than the entropy functional used in the phase field method.”

Additionally, Equation (15) uses the method of Lagrange multipliers (https://en.wikipedia.org/wiki/Lagrange_multiplier) which is a common tool in the machine learning field (e.g. maximum entropy principle) for optimization. We plan to indicate the name of the method (method of Lagrange multiplier) for readers to understand the physical interpretation.

We will also address all points in the next stage.

References

Cahn, J. W., & Hilliard, J. E. (1958). Free energy of a nonuniform system. I. Interfacial free energy. The Journal of chemical physics, 28(2), 258-267.

Cassel, K. W. (2013). Variational methods with applications in science and engineering. Cambridge University Press.

Singer-Loginova, I., & Singer, H. M. (2008). The phase field technique for modeling multiphase materials. Reports on progress in physics, 71(10), 106501.